# A supplementary material for Lecture 8

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In this note, we prove the validity of the Gibbs sampler (Example 6.14.6 of PRP).

## 1 Description of the Gibbs sampler and its validity

Let $\Theta = S^V$, where $S$ is a finite (or countable) set called the 'local state space' and $V$ is a finite set called the 'index set'. Suppose that we want to generate random samples from

$$\pi_i = \frac{\pi_i^*}{Z}, \quad i = (i_v)_{v \in V} \in \Theta,$$

where $\pi_i^*$ is a given positive function[*1] and $Z = \sum_{i \in \Theta} \pi_i^* < \infty$.

### 1.1 The Gibbs sampler for a fixed index

Fix an index $v \in V$. For each $i \in \Theta$, let

$$\Theta_{i,v} = \{j \in \Theta \ : \ j_w = i_w \text{ for } w \neq v\}.$$

Define

$$h_{ij} = \frac{\pi_j}{\sum_{k \in \Theta_{i,v}} \pi_k} = \frac{\pi_j^*}{\sum_{k \in \Theta_{i,v}} \pi_k^*}, \quad j \in \Theta_{i,v}.$$

The *Gibbs sampler* for the fixed index $v$ is a Markov chain $\{X_n\}_{n \geq 0}$ described as follows.

- If the current state is $X_n = i$, then $X_{n+1} = j$ is generated according to the probability mass function $(h_{ij})_{j \in \Theta_{i,v}}$.

Denote the transition matrix of the Markov chain by $\boldsymbol{P}^v = (p_{ij}^v)$. Note that $\boldsymbol{P}^v$ is *not* irreducible (except for some trivial cases) since $v$ is fixed.

**Lemma 1.** The transition matrix $\boldsymbol{P}^v$ satisfies the detailed balance equation with respect to $\boldsymbol{\pi} = (\pi_i)_{i \in \Theta}$.

[*1] If the function $\pi_i^*$ is allowed to take the zero vaule, it will need additional care about irreducibility (see Lemma 2 and Lemma 3).

*Proof.* The transition matrix is given by $p^v_{ij} = h_{ij}$ if $j \in \Theta_{i,v}$, and $p_{ij} = 0$ otherwise. If $j \in \Theta_{i,v}$, then $\Theta_{j,v} = \Theta_{i,v}$ and therefore

$$\pi_i p^v_{ij} = \frac{\pi_i \pi_j}{\sum_{k \in \Theta_{i,v}} \pi_k} = \frac{\pi_i \pi_j}{\sum_{k \in \Theta_{j,v}} \pi_k} = \pi_j p^v_{ji}.$$

If $j \notin \Theta_{i,v}$, then $i \notin \Theta_{j,v}$ and therefore

$$\pi_i p^v_{ij} = 0 = \pi_j p^v_{ji}.$$

Hence the detailed balance equation holds. □

It is necessary to modify the Markov chain in order to make it irreducible. As described in Example 6.14.6 of PRP, we may choose the index $v$ at random or in pre-determined manner. We discuss the two methods in the following subsections.

## 1.2 The Gibbs sampler with the randomized index

If the index $v$ is randomly chosen, the resultant chain has the transition matrix

$$\boldsymbol{P} = \frac{1}{|V|} \sum_{v \in V} \boldsymbol{P}^v.$$

**Lemma 2.** The matrix $\boldsymbol{P}$ is irreducible and satisfies the detailed balance equation.

*Proof.* The detailed balance equation of $\boldsymbol{P}$ follows from that of $\boldsymbol{P}^v$. To check the irreducibility, let $i$ and $j$ be any states in $\Theta$. Assign numbers of the elements of $V$ as $V = \{1, \ldots, n\}$ without loss of generality. Define $i^{(0)}, i^{(1)}, \ldots, i^{(n)}$ by

$$i^{(k)}_v = \begin{cases} j_v & \text{if } v \le k, \\ i_v & \text{if } v > k. \end{cases}$$

for $0 \le k \le n$. Then we have $i^{(0)} = i$, $i^{(n)} = j$, and

$$p_{i^{(k-1)} i^{(k)}} \ge \frac{1}{|V|} p^k_{i^{(k-1)} i^{(k)}} > 0$$

for $1 \le k \le n$. Hence $\boldsymbol{P}$ is irreducible. □

## 1.3 The Gibbs sampler with the pre-determined index

Let $V = \{1, \ldots, n\}$ without loss of generality. If we apply the Markov chain $\boldsymbol{P}^v$ for $v = 1, \ldots, n$ in order, then the resultant Markov chain is given by

$$\boldsymbol{P} = \boldsymbol{P}^1 \cdots \boldsymbol{P}^n.$$

In general, $\boldsymbol{P}$ does not satisfy the detailed balance equation. But the following lemma holds.

**Lemma 3.** The matrix $\boldsymbol{P}$ is irreducible and has the stationary distribution $\boldsymbol{\pi}$.

*Proof.* Irreducibility is proved in the same manner as Lemma 2. For stationarity, we have

$$\begin{aligned}\boldsymbol{\pi}\boldsymbol{P} &= \boldsymbol{\pi}\boldsymbol{P}^1 \cdots \boldsymbol{P}^n \\ &= \boldsymbol{\pi}\end{aligned}$$

since $\boldsymbol{\pi}$ is a stationary distribution of each $\boldsymbol{P}^v$. □