2nd International Symposium on
Information Geometry and its Applications
December 12-16, 2005, Tokyo

STATISTICAL MECHANICS OF HARMONIC OSCILLATORS ON RIEMANNIAN MANIFOLDS

YUKIO OTSU

Let $X = (X, d)$ be a compact n-dim Riemannian manifold, where d is the metric induced from the Riemannian metric. Fix a net $\Lambda = (p_0, \ldots, p_M)$, i.e., an ordered set of $M+1$ points of X. Put

$$a_{ij} = \begin{cases} 1 & \text{if } d(p_i, p_j) < r \\ 0 & \text{otherwise} \end{cases}, \qquad d_i = \sum_{j=0}^{M} a_{ij}$$

for $r > 0$ and define the discrete Laplacian $L = L_{r,\Lambda}$ by

$$L_{ij} = \frac{\text{const}}{r^{n+2}M}(d_i \delta_{ij} - a_{ij}).$$

Since $L_{r,\Lambda}$ is nonnegative symmetric with respect to the inner product

$$((c_i), (e_i))_M = \frac{\text{const}}{M} \sum_{i=0}^{M} c_i e_i \qquad \text{for } (c_i), (e_i) \in \mathbf{R}^{M+1},$$

its eigenvalues are written as

$$\omega_0^2(r, \Lambda) = 0 \leq \omega_1^2(r, \Lambda) \leq \cdots \leq \omega_i^2(r, \Lambda) \leq \cdots \leq \omega_M^2(r, \Lambda).$$

We can show that $\{L_{r,\Lambda}\}$ converges to the continuum Laplacian Δ of X for "random net" Λ, in particular, $M \to \infty$ and $r \searrow 0$, in certain statistical sense.

Let $\tilde{\mathcal{X}}^M$ be the set of pairs of compact n-dim Riemannian manifold and its net. Note that we can introduce an "affine" structure on $\tilde{\mathcal{X}}^M$ for $L_{r,\Lambda}$ is an $(M+1) \times (M+1)$ matrix. To investigate the structure of $\tilde{\mathcal{X}}^M$ we use the method of quantum statistical mechanics. For $i = 1, \ldots, I \leq M$ define generalized coordinate and momentum operators $\hat{Q}_i$, $\hat{P}_i$ on $L^2(\mathbf{R}^I)$ by Schröedinger's representation. For each $L_{r,\Lambda}$ we define the Hamiltonian of harmonic oscillators

$$\hat{H} = \hat{H}_{r,\Lambda,I} = \tfrac{1}{2} \sum_{i=1}^{I} (\hat{P}_i^2 + \omega_i^2(r, \Lambda) \hat{Q}_i^2) : L^2(\mathbf{R}^I) \to L^2(\mathbf{R}^I)$$

and its (discrete) free energy $F_{r,\Lambda,I}(b, N) = \log \operatorname{tr}(1 - b\hat{H})^N$ for $b > 0$ and $N \in \mathbf{N}$. In this way we get a function $\tilde{\mathcal{X}}^M \ni (X, \Lambda) \mapsto F_{r,\Lambda,I}(b, N) \in \mathbf{R}$. By interpreting it as a potential function we can define a "Hessian metric" on $\tilde{\mathcal{X}}^M$. In this talk, we examine the limit of these metrics (on the space of Riemannian manifolds) in the case where $N \to \infty$, $b \searrow 0$ and random net Λ under the conditions $Nb \to \beta > 0$ and $r = b\hbar$ for fixed $\hbar > 0$.