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A QUANTUM/COMPLEX EXTENSION OF INFORMATION GEOMETRY

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Let $\mathcal{H} \cong \mathbb{C}^d$ be a Hilbert space. Introducing the equivalence relation $\sim$ on $\mathcal{H} \setminus \{0\}$ by $\xi_1 \sim \xi_2 \overset{\text{def}}{\Longleftrightarrow} \exists c \in \mathbb{C},\ \xi_1 = c\,\xi_2$, the complex projective space is defined as the quotient space: $\mathbb{P}(\mathcal{H}) = (\mathcal{H} \setminus \{0\})/\sim$, which is a $(d-1)$-dimensional complex manifold. We can identify it with the set of pure states in quantum mechanics: $\mathbb{P}(\mathcal{H}) = \{|\xi\rangle\langle\xi| \mid \xi \in \mathcal{H},\ \|\xi\| = 1\}$.

The inner product on $\mathcal{H}$ naturally induces a Riemannian metric g on $\mathcal{H}$, which is called the Fubini-Study metric. Here we redefine g by multiplying the standard definition by 4. From a viewpoint of quantum estimation theory, g can be regarded as an analogue of Fisher information metric as follows. Let $[\theta^i]$ be a local coordinate system on $\mathbb{P}(\mathcal{H})$, whereby an element of $\mathbb{P}(\mathcal{H})$ is parametrized as ρ_θ. The components of g are then represented as $g_{ij} = \mathrm{Re}\ \mathrm{Tr}\,[\rho_\theta L_{\theta,i} L_{\theta,j}]$, where $L_{\theta,i}$ is a hermitian operator on $\mathcal{H}$ satisfying $\frac{\partial}{\partial\theta^i}\rho_\theta = \frac{1}{2}\left(\rho_\theta L_{\theta,i} + L_{\theta,i}\rho_\theta\right)$. The operators $\{L_{\theta,i}\}$ and the matrix $[g_{ij}]$ are called symmetric logarithmic derivatives (SLDs) and the SLD Fisher information matrix, respectively, and play important roles in quantum estimation theory.

From the fact that $\mathbb{P}(\mathcal{H})$ is a complex manifold, a $(1,1)$-tensor field J satisfying $J^2 = -1$ is canonically defined, and a skew-symmetric bilinear form (differential 2-form) ω is defined by $\omega(u,v) = g(u, Jv)$. It is well known that g is a Kähler metric in the sense that $d\omega = 0$.

Suppose that we are given a point $\rho_0 \in \mathbb{P}(\mathcal{H})$ and hermitian operators $\{F_1, \ldots, F_n\}$ on $\mathcal{H}$ such that $F_i F_j = F_j F_i$ $(\forall i, j)$ and that $\{\rho_0, F_1\rho_0, \ldots, F_n\rho_0\}$ are linearly independent (which implies $n \le d-1$). For $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$ with $z^i = \theta^i + \sqrt{-1}y^i$ $(\theta^i, y^i$: real), let

$$\rho_z := \exp\Big[\frac{1}{2}\Big(\sum_i z^i F_i - \psi(\theta)\Big)\Big]\rho_0 \exp\Big[\frac{1}{2}\Big(\sum_i \bar{z}^i F_i - \psi(\theta)\Big)\Big],$$

where $\bar{z}^i = \theta^i - \sqrt{-1}y^i$. Properly choosing a neighborhood V of $\mathbb{R}^n$ in $\mathbb{C}^n$ so that $V \ni z \mapsto \rho_z$ is injective, the set $M := \{\rho_z \mid z \in V\}$ becomes a complex submanifold of $\mathbb{P}(\mathcal{H})$ with holomorphic coordinates $[z^i]$, on which the Kähler structure (J, g, ω) is inherited from $\mathbb{P}(\mathcal{H})$. On the other hand, if the coodinates are restricted to real numbers so that $z^i = \theta^i$, we obtain $N := \{\rho_\theta \mid \theta \in V \cap \mathbb{R}^n\}$, which is in the form of quasi-classical exponential family and is dually flat with respect to $(g, \nabla^{(\mathrm{e})}, \nabla^{(\mathrm{m})})$, where g is the Fubini-Study (= SLD Fisher) metric on N, $\nabla^{(\mathrm{e})}$ and $\nabla^{(\mathrm{m})}$ are flat affine connections with affine coordinates $[\theta^i]$ and $[\eta_i] := [\mathrm{tr}\,(\rho_\theta F_i)]$. The dually flat structure and the Kähler structure are closely related. Remarkable manifestations are: $\omega = \sum_i d\eta_i \wedge dy^i$, $\nabla^{(\mathrm{e})} \circ J = J \circ \nabla^{(\mathrm{m})}$ and $\nabla^{(\mathrm{e})}\omega = \nabla^{(\mathrm{m})}\omega = 0$, where the e, m-connections are properly extended to M. In addition, $4\psi(\theta)$ gives a Kähler potential of M. In my talk at the symposium, I will explain some details of these structures as well as their implications to quantum estimation theory and related subjects.